Tipping points, bifurcations and instabilities of nonautonomous systems

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Tipping points and bifurcations

Henri Poincaré taught us a great deal about how to understand the dynamical behaviour of models given by autonomous ODEs:

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is some (possibly high dimensional) state space. We are usually interested in solving an initial value problem where $x(0) = x_0$. He discovered that if $f(x)$ is a nonlinear function of state then the solution $x(t)$ can have a sensitive dependence on x_0.

A general nonlinear ODE may be *multistable*, i.e. have several different identifiable asymptotic states that "typical" solutions of the IVP will converge towards; these are the *attractors* of the system.

It turns out to be helpful to explore parameter-dependent solutions of *bifurcation problems*:

$$\dot{x} = f(x, \lambda), \tag{2}$$

where λ is a (vector of) real parameters.

Typical attractors are robust, i.e. will persist for small changes in λ, but at *bifurcation points* the set of attractors changes in a qualitative way.

In general, one cannot find solutions $x(t)$ of nonlinear differential equations

$$\dot{x} = f(x)$$

for $x \in \mathbb{R}^n$ and even quite simple functions f with n small. Possible ways forward are:

- Option 1: Use numerical approximation.
- Option 2: Find simple solutions (equilibria, periodic orbits) and determine their stability.

Unfortunately option 1 may give much data but not give much insight, and option 2 may not tell us about the 'typical solutions' that we want to know about.

- Local bifurcation theory deals with *equilibria* (also known as *steady solutions* or *singular points*), i.e. x_0 such that

$$f(x_0, \lambda) = 0.$$

- Equilibria typically come in *branches* i.e. $(X(\lambda), \lambda)$ parametrized by the bifurcation parameter. There may be many branches at any given λ.
- We usually express the bifurcation pattern in a *bifurcation diagram* which plots some measure of the solution x (vertical axis) against the parameter (horizontal axis). A branch is plotted as a smooth line on such a diagram.
- Typical choices for the vertical axis are: one of the coordinates of x; a norm of x but any smooth observable of x may be shown.

Linear stability of equilibrium $(X, 0)$. can be found by examining $J = Df(X, 0)$ the Jacobian of equilibrium solution:

- If no eigenvalues of J are on the imaginary axis, then we say X is *hyperbolic* and $(X, 0)$ is a point on a branch of equilibria.
- If at least one eigenvalues of J are on the imaginary axis, then we say X is at a *bifurcation* and more than one branch may meet at $(X, 0)$.

For typical choice of f the **only generic bifurcations** are following:

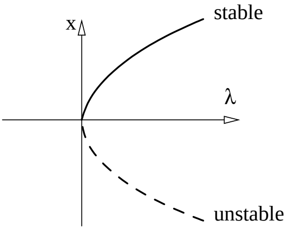
- *Saddle-node bifurcation* where there is a single zero eigenvalue of J.
- *Hopf bifurcation* where there is a single pure imaginary pair of eigenvalues of J.

Saddle-node bifurcation

Normal form of a saddle-node bifurcation in one dimension

$$\dot{x} = \lambda - x^2 \tag{3}$$

Two equilibria for $\lambda > 0$, one for $\lambda = 0$ and none for $\lambda < 0$.



Bifurcation diagram

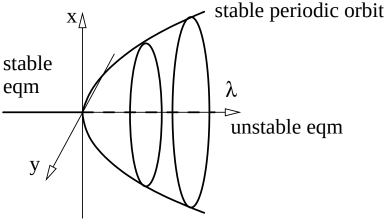
The same bifurcation diagram (up to reflection in x and/or λ) holds for ALL saddle-node bifurcations. Equation (3) is the *normal form* for a saddle-node bifurcation.

Hopf bifurcation

Normal form of a Hopf bifurcation in two dimensions $z = x + iy$:

$$\dot{z} = (\lambda + i\omega)z - |z|^2 z \quad (4)$$

One equilibrium for all λ. A *periodic orbit* appears on increasing λ through zero.



Bifurcation diagram
The same bifurcation diagram (up to coordinate changes) holds for ALL Hopf bifurcations. Equation (4) is the *normal form* for a Hopf bifurcation.

What is meant by a bifurcation problem being *generic*?

There is always an implied context for a system, for instance if

$$\dot{x} = f(x, \lambda)$$

is a model of a simple physical system with $x, \lambda \in \mathbb{R}$ then we hope that the predictions of the model are not sensitive to small details in the specification of f.

Suppose that $f \in C^\infty(\mathbb{R}, \mathbb{R})$ is a particular function, then unless we have a compelling reason to believe that f must have a special property (such as oddness $f(-x) = -f(x)$), we assume that is has no such property.
Formally, let $P(f)$ be some property of a $f \in C^\infty$.

- We say the property P is *generic* if it holds on an open dense subset of C^∞.
- Otherwise we say it is *non-generic*.

Genericity depends on *context*!

Let A be the set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$.

Let $A_s \subset A$ be the set of all symmetric (i.e. odd) smooth functions $f : \mathbb{R} \to \mathbb{R}$.

- It is not generic for $f \in A$ to be also in A_s, i.e. to be an odd function.
- It is generic for $f \in A$ to be non-constant.
- It is generic for $f \in A_s$ to have a simple zero at $x = 0$.
- It is not generic for $f \in A$ to have any zero.
- BUT: It is not generic for $f \in A$ to have no zero.

If we assume there is no special structure/symmetry, the only bifurcations are saddle-node and Hopf. If however we know there are special structures in the model, there may be other bifurcations that become generic

Instabilities of nonautonomous systems

"Autonomous systems are all alike; each nonautonomous system is nonautonomous in its own particular way."

Many applications require an understanding the dynamical behaviour of nonautonomous systems

$$\dot{x} = f(x, \Lambda(rt)), \tag{5}$$

However, many methods are only applicable to autonomous systems of the form

$$\dot{x} = f(x, \lambda). \tag{6}$$

Clearly if $r > 0$ is small then we expect (6) to give a lot of information about (5). In particular we expect bifurcations to be key for small $r > 0$. Other effects such rate-induced tipping can appear for larger r.

For nonautonomous systems, various types of instability can independently occur when a dynamical system is subjected to time-varying inputs. This includes:

- Bifurcation-induced tipping (B-tipping) in response to slowly varying inputs.
- Noise-induced tipping (N-tipping) in response to large deviations in noise.
- Rate-induced tipping (R-tipping) when time-variation of input parameters of a dynamical system interacts with system timescales to give genuine nonautonomous instabilities.

Such instabilities appear as the input varies at some critical rates and cannot, in general, be understood in terms of autonomous bifurcations in a frozen system.

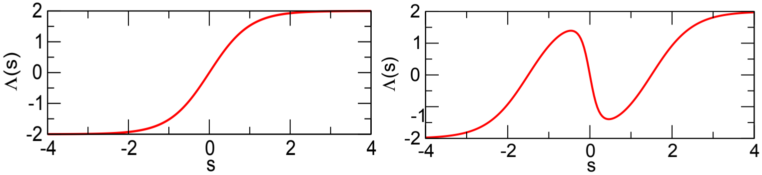
In typical situations more than one type of tipping effect may be present.

[A, Cox, Vitolo, Wieczorek 2012]

We say Λ is a *parameter shift* if

$$\Lambda(s) \to \lambda_{\pm}$$

as $s \to \pm\infty$.



Examples of $\Lambda(s) \in \mathcal{P}(-2, 2)$.

A solution $x(t)$ of the nonautonomous system is a (point, local) *pullback attractor* if there is a bounded open set $U \subset \mathbb{R}^n$ with the following property: for any $y \in U$ and $t \in \mathbb{R}$

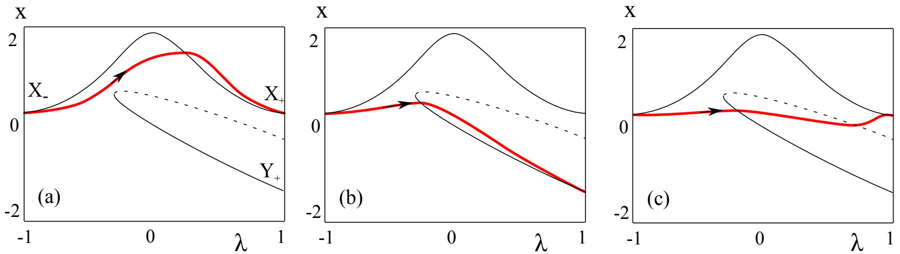
$$|\Phi_{t,t-s}(y) - x(t)| \to 0 \text{ as } s \to \infty$$

where $\Phi_{t,u}(y)$ is the solution evolved from being at location y at time u forward to time $t > u$.

If a $x(t)$ is a stable solution for λ_-, there is a there is a unique trajectory $\tilde{x}_{pb}(t)$ with

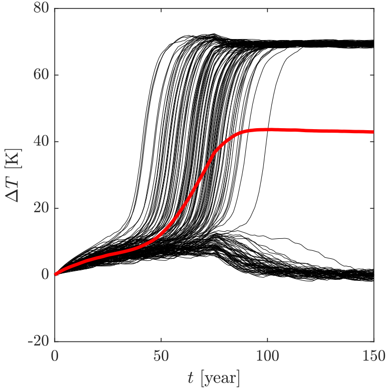
$$\tilde{x}_{pb}(t) \to X_- \text{ as } t \to -\infty.$$

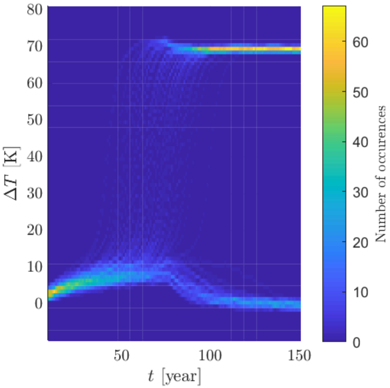
[Chekroun, Simonnet and Ghil,2011]
[A, Perryman and Wieczorek, 2017]



Bifurcation diagrams showing stable (black solid lines) and unstable (black dashed lines) equilibria for a system with λ between $\lambda_- = -1$ and $\lambda_+ = +1$. The red lines show the pullback attractor starting at X_- for parameter shifts on varying the rate r. (a) For $r < r_1$ the pullback attractor end-point tracks the branch through to X_+. (b) There is a range of rates $r_1 < r < r_2$ where the pullback attractor R-tips to Y_+. (c) For $r > r_2$ the pullback attractor again end-point tracks the branch through to X_+.

Other examples of rate-dependent effects, e.g. partial tipping of an ensemble:





An ensemble of 150 runs of an energy balance climate model with chaotic forcing and instantaneous albedo relaxation. An abrupt CO_2 quadrupling is applied for 75 years, after which the initial CO_2-level is restored. [Bastiaansen, A & von der Heydt 2023]

Example: tipping between chaotic attractors

The double scroll circuit, introduced by Chua *et al.* is

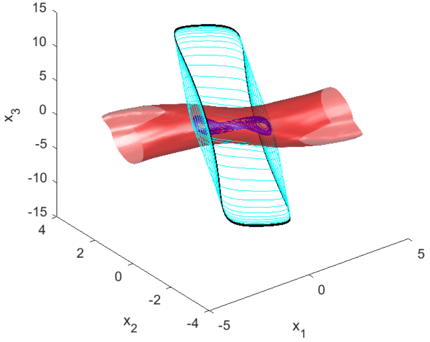
$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2, x_3) := a(x_2 - \phi(x_1))\\ \dot{x}_2 &= F_2(x_1, x_2, x_3) := x_1 - x_2 + x_3\\ \dot{x}_3 &= F_3(x_1, x_2, x_3) := -bx_2\end{aligned} \tag{7}$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, with

$$\phi(x_1) = \frac{1}{16}x_1^3 - \frac{1}{6}x_1.$$

We choose parameters where there is bistability between two attractors: a "double scroll" chaotic attractor, and a large-amplitude limit cycle.

[A & Newman 2021]



Double scroll system (7) showing a chaotic attractor A_1 (purple) enclosed within a tube-like basin of attraction with boundary shown in red. An initial condition outside the basin approaches (cyan) a large amplitude limit cycle A_2 (black).

Movie

Consider double scroll system with shift

$$\dot{x} = F\left(x - \Lambda(rt)(1,1,0)\right) \tag{8}$$

where

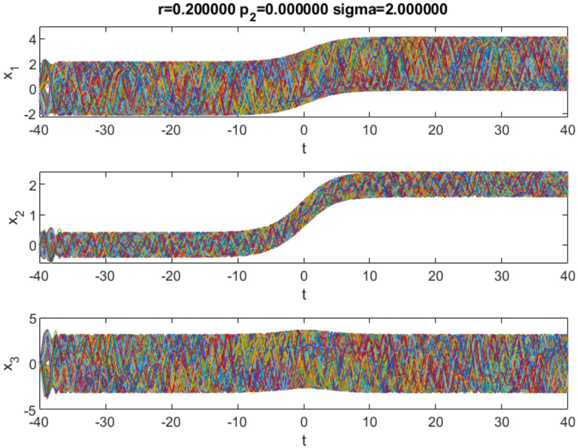
$$\Lambda(t) = \frac{\sigma}{2}\left(1 + \tanh(t)\right). \tag{9}$$

In the limit $r \gg 1$, (8) has discontinuous right hand side:

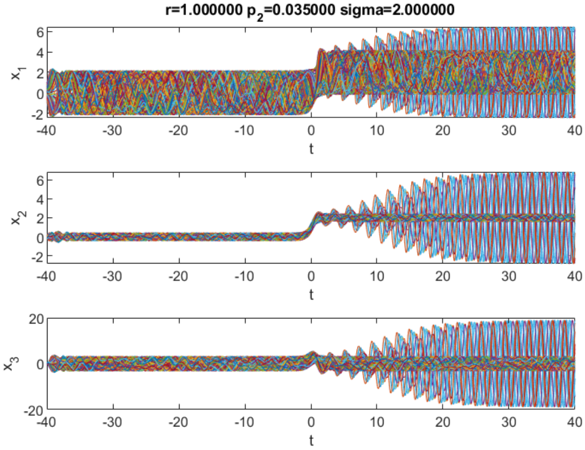
$$F(x - \Lambda(rt)(1,1,0)) = \begin{cases} f^{-}(x) & \text{for } t < 0 \\ f^{+}(x) & \text{for } t > 0 \end{cases} \tag{10}$$

where

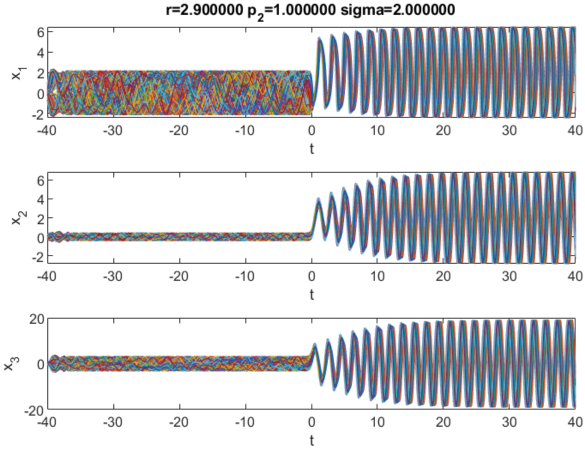
$$f^{-}(x) = F(x), \quad f^{+}(x) = F(x - (\sigma, \sigma, 0)).$$



Numerics: many trajectories starting in basin of past limit attractor at $T = -40$ (approximation of physical measure) for $r = 0.82$;
Tracking



Numerics (approximation of physical measure) for $r = 1$;
Partial Tipping



Numerics (approximation of physical measure) for $r = 2.98$:
Total Tipping

Parameter shifts involving chaotic attractors

Write the solution $x(t)$ of the nonautonomous system

$$\dot{x} = f(x, \Lambda(rt)), \tag{11}$$

with $x(s) = x_0$ and $t > s$ as a process

$$x(t) = \Phi^{(r)}(t, s, x_0)$$

for any $t > s$. We consider a *parameter shift* $\Lambda(t)$ where $\lim_{t\to\pm\infty} = \lambda^{\pm}$.

For the frozen system

$$\dot{x} = f(x, \lambda), \tag{12}$$

we can write the solution as a flow $x(t) = \varphi_{\lambda}(t - s, x(s))$ for any $t > s$ and fixed λ; we write $\varphi_{\lambda^{\pm}}$ for the future/past limit systems respectively.

For each asymptotically stable attractor A_- for the past limit system there is a local pullback attractor for (11) whose (upper) backward limit is contained in A_- [Alkhhayuon & A 2018].

We construct a pullback attractor by:

$$A_t^{[\Lambda,r,A_-]} := \bigcap_{\tau>0} \overline{\bigcup_{s\leq\tau} \Phi(t,s,\mathbb{N}_\eta(A_-))} \tag{13}$$

for small $\eta > 0$. If A_- is an equilibrium then shows that the pullback attractor is a single trajectory or so-called *pullback attracting solution*.

For a uniformly exponentially stable branch $A(\lambda)$ that contains an attractor of the past limit system $A_- := A(\lambda_-)$ and for sufficiently small positive r, the pullback attractor end-point tracks the branch to $A(\lambda_+)$.

This tracking is not guaranteed for large values of $r > 0$ or where a stable branch is weakened to a stable path. Rate-induced transitions take place when this tracking breaks.

[Alkhayuon & A 2018] investigate A_- periodic attractors and note there is a rate-induced transition from tracking to partial tipping and another to total tipping.

Nonautonomous Physical Measures

An *empirical measure* μ_{T,x_0} for an autonomous flow φ is defined for any measurable set S as

$$\mu_{T,x_0}(S) = \frac{\ell(\{t \in [0,T] : \varphi(t,x_0) \in S\})}{T}.$$

This can be used to define "natural" or "physical" measures as a limit point of such measures:

Given an attractor A of an autonomous ODE with basin of attraction B, a *physical measure on* A is a probability measure μ with $\mathrm{supp}(\mu) = A$ such that for Lebesgue-almost every $x_0 \in B$, as $T \to \infty$ the empirical measure μ_{T,x_0} converges weakly to μ.

To generalize to a nonautonomous setting, [A & Newman 2021] use a stronger notion of attraction:

Given an attractor A of (6) with basin of attraction B, an *"attracting measure"* is a physical measure μ supported on A such that for every probability measure ν_0 absolutely continuous w.r.t. Lebesgue where the density $h \in L^1(\mathbb{R}^d)$ is supported within B, then

$$\nu_T(S) := \int_S (\mathcal{L}_{\varphi(T)} h)(y)\, dy = \int_{\mathbb{R}^d} \mathbb{1}_S(\varphi(T, x)) h(x)\, dx$$

converges weakly to μ as $T \to \infty$.

We define nonautonomous attractors and physical measures relative to a given local attractor A^- of the past limit system (with basin B^-):

For the nonautonomous system (5), a *pullback attractor starting at the attractor A^- of the past limit* is a nonautonomous invariant set $\mathcal{A} = \{A(t)\}$ such that:

1. $\bigcup_{t\in\mathbb{R}} A(t)$ is bounded, and for each $t \in \mathbb{R}$, $A(t)$ is closed;
2. for any bounded neighbourhood U of A^- with $\bar{U} \subset B^-$, for each $t \in \mathbb{R}$ and $\varepsilon > 0$, taking sufficiently large-magnitude $s < 0$ gives

$$A(t) \subset \Phi^{(r)}(t, s, U) \subset B_\varepsilon(A(t)).$$

We define the *nonautonomous empirical measure*:

$$\mu_{\tau,\tau-T,x_0}(S) = \frac{\ell(\{t \in [\tau - T, \tau] : \Phi^{(r)}(\tau, \tau - T, x_0) \in S\})}{T}.$$

Given an attractor A^- for the past limit system and a pullback attractor $\mathcal{A}$ starting at A^-, a *physical measure on* $\mathcal{A}$ is a probability measure μ on $\mathcal{A}$ such that $\mathrm{supp}(\mu_t) = \mathcal{A}(t)$ at each $t \in \mathbb{R}$ and for Lebesgue-almost every $x_0 \in B$, the empirical measure $\mu_{t,t-T,x_0}$ converges weakly to μ_t a $T \to \infty$.

A *pullback-attracting measure on* $\mathcal{A}$ is a probability measure μ on $\mathcal{A}$ such that $\mathrm{supp}(\mu_t) = \mathcal{A}(t)$ at each $t \in \mathbb{R}$ and for every probability measure ν_0 of smooth density h supported within B, for each $t \in \mathbb{R}$, as $T \to \infty$ the measure $\nu_T^t := \Phi^{(r)}(t, t - T, \nu_0)$ converges weakly to μ_t.

In [Newman & A 2023] we give sufficient conditions on such a scenario that pullback attracting physical measures exist. These involve assuming

(a) Mixing properties of the past limit measure so that the physical measure attracts other a.c. measures supported within the basin and

(b) Exponential convergence properties of the measures (and their support) in the limit $T \to \infty$.

In [Newman & A 2023] we first start in the autonomous setting (applied to the past-limit system Ψ_*^t acting on Borel probability measures with Wasserstein distance d_W). We say that μ is:

- an *attracting measure of the past-limit system* if there exists a neighbourhood $U \subset X$ of P such that for every Lebesgue-absolutely continuous probability measure λ with $\lambda(U) = 1$, λ is attracted to μ under (Ψ_*^t);
- a *physical measure of the past-limit system* if there exists a neighbourhood $U \subset X$ of P such that for Lebesgue-almost all $x \in U$, δ_x is Cesàro-attracted to μ under (Ψ_*^t);

Suppose μ is an attracting measure of the past limit system with support P. An orbit (μ_t) of $(\Phi_{s,t*})$ is called

- an *attracting measure rooted at* μ if there exists a neighbourhood $U \subset X$ of P such that for every Lebesgue-absolutely continuous probability measure λ with $\lambda(U) = 1$, λ is pullback-attracted to (μ_t) under $(\Phi_{s,t*})$;
- a *physical measure rooted at* μ if there exists a neighbourhood $U \subset X$ of P such that for Lebesgue-almost all $x \in U$, δ_x is pullback-Cesàro-attracted to (μ_t) under $(\Phi_{s,t*})$.

Suppose $\nu(\lambda)$ is a family of invariant measures for the frozen systems with support $Q(\lambda)$ that limit to an attracting measure μ with support P for $\Lambda(t) \to \lambda_{-\infty}$ as $t \to -\infty$.

Theorem

(A) *Suppose* $d_W(\nu(\Lambda(t)), \mu) = o(e^{-r|t|})$ *and* $d_H(Q(\Lambda(t)), P) = o(e^{-r|t|})$ *as* $t \to -\infty$. *Then there exists an orbit* (μ_t) *of* $(\Phi_{s,t*})$ *such that* μ *is pullback-attracted to* (μ_t) *under* $(\Phi_{s,t*})$.

(B) *Suppose, moreover, that* $\max_{x\in X} |f(.,\Lambda(t)) - f(.,\lambda_{-\infty})| = o(e^{-r|t|})$ *and* P *is Lyapunov-stable under the past-limit system. If* μ *is an attracting measure (resp. physical measure, weakly physical measure) of the past-limit system, then* (μ_t) *is an attracting measure (resp. physical measure, weakly physical measure) rooted at* μ.

[Newman & A 2023]

Strategy of proof:

For (A), we show that there exists a closed neighbourhood O of P and a value $T^* \leq 0$ such that, writing $\mathcal{O} := \{\lambda \in M_X : \lambda(O) = 1\}$, $(\mathcal{O}, T^*, \nu)$ is a "Monotone-like Nonautonomousness Controller" (MLNAC) of $(M_X, d_W, (\Phi_{s,t*}))$ that gives bounds near μ of rate r and $(\mathcal{O}, T^*)$ is a "Growth Controller" (GC) of $(M_X, d_W, (\Phi_{s,t*}))$ of rate r; this can be used to give the desired estimates.

For (B), we show that there exists a neighbourhood U of P such that for every $\lambda \in M_X$ with $\lambda(U) = 1$,

- λ is pullback-eventually in $\mathcal{O}$ under $(\Phi_{s,t*})$;
- if λ is attracted (resp. Cesàro-attracted) to μ under (Ψ_*^t), then λ is past-attracted (resp. past-Cesàro-attracted) to (μ_t) under $(M_X, d_W, (\Phi_{s,t*}))$ with nonautonomous error of decay rate r.

Tipping between past and future attractors

Given a number of past and future attractors, physical measures allow one to define a notion of probability of tipping between these attractors. Suppose that:

A1 The past limit system has an attractor A^-

A2 The nonautonomous system admits a pullback attractor $\mathcal{A}$ starting at A^-, and there is a physical measure μ on this pullback attractor.

A3 The future limit system has disjoint attractors $A_1^+, \ldots, A_{n_+}^+$ with basins of attraction $B_1^+, \ldots, B_{n_+}^+$ that exhaust Lebesgue measure.

A4 The A_j^+ are Lyapunov stable.

A5 The physical measure "does not get caught" on the boundaries of B_j^+.

Theorem

Consider the assumptions as above. Then

(A) *For each $j \in \{1, \dots, n_+\}$, the limit*

$$p_j := \lim_{t\to\infty} \mu_t(B_j^+)$$

exists, and for every neighbourhood U of A_j^+ with $U \subset B_j^+$ we have

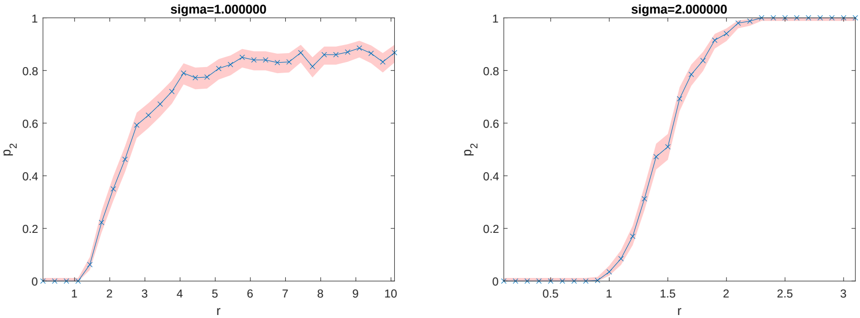
$$p_j = \lim_{t\to\infty} \mu_t(U).$$

(B) *We have $\sum_{j=1}^{n_+} p_j = 1$.*

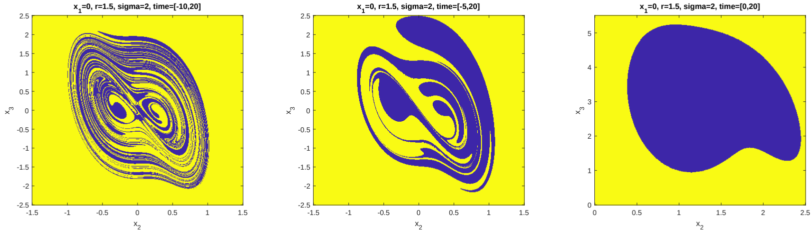
We call the value p_j the *probability of tipping from A^- to A_j^+* for each $j \in \{1, \dots, n_+\}$.

[A & Newman 2021]

Returning to the Double scroll with parameter shift:

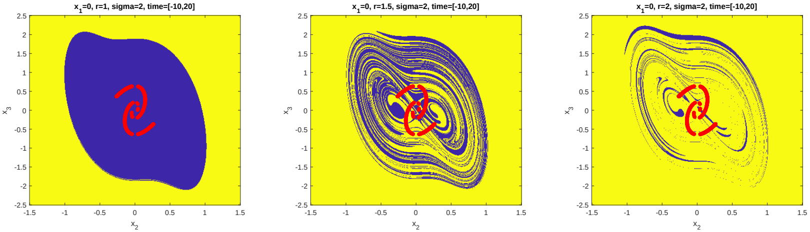


Tipping probability from chaotic A_1^- to periodic A_2^+ attractor for the double scroll system with parameter shift. (Left) $\sigma = 1$: asymptotically fast shift still gives partial tipping, corresponding to the attractor A_1^- partially intersecting the basin of A_1^+. (Right) $\sigma = 2$.



Points in $x_1 = 0$ at time T that are asymptotic to the future chaotic attractor A_1^+ (blue) and the future periodic attractor (yellow), for the double scroll system with parameter shift for $\sigma = 2$ and $r = 1.5$.

Panels correspond to $T = -10$, $T = -5$ and $T = 0$ respectively.



Points in $x_1 = 0$ at time $T = -10$ asymptotic to the future chaotic attractor A_1^+ (blue) and the future periodic attractor (yellow) for different rates (left) $r = 1$, (mid) $r = 1.5$, (right) $r = 2$.

The red points show a section through the pullback attractor at this time.

Discussion and future challenges

Some challenges for the future:

(a) Gain a better theoretical understanding of nonautonomous physical measures.
(b) Understand cases where a physical measure splits into the basin of several future attractors.
(c) Understand thresholds and edge states in cases where it splits.
(d) Understand implications for multiscale chaotic systems.
(e) Use these insights to improve the science of ensemble forecasts.

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